MATH 1010E University Mathematics
Lecture Notes (week 2)
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1 More about exponential function

Last week, we have defined the exponential function exp: $\mathbb{R} \to \mathbb{R}$ to be

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

You may have also learned about Euler's number e, which is an irrational number defined by the "limit"

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

We can therefore define the "exponential function with base e" as the function $x \mapsto e^x$. We show that it indeed agrees with the exponential function exp defined by an infinite series.

Theorem 1.1 *$e^x = exp(x)$ for all $x \in \mathbb{R}$.*

Proof: We first show that it is true for $x = 1$, i.e. $\exp(1) = e$, or equivalently,

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

Using the binomial theorem,

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right). \end{aligned}$$

Taking $n \to \infty$, the left hand side approaches e and the right hand side approach $\exp(1)$ since each term in the bracket approach 1 as $n \to \infty$. Hence, we have shown that $e = \exp(1)$.

To show that $e^x = \exp(x)$ for all $x \in \mathbb{R}$, assuming we can move the "lim" outside the square bracket below:

$$e^x = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}.$$

Multiplying both the numerator and denominator of $\frac{1}{n}$ by x and let $m := nx$, noting that $m \to \infty$ as $n \to \infty$, the right hand side becomes

$$\lim_{n\to\infty}\left(1+\frac{x}{nx}\right)^{nx} = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m.$$

Using the binomial theorem again and argue as before, we can show that the right hand side is

$$\lim_{m\to\infty}\sum_{k=0}^{m}\binom{m}{k}\frac{x^k}{m^k} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \exp(x).$$

This proves the theorem.

Question: Try to catch the "loopholes" in the proof above. Can you give a more rigorous proof?

Now, let's look at more properties of the exponential function e^x.

Proposition 1.2 *The following statements hold:*

(i) $e^{x+y} = e^x \cdot e^y$ *for all* $x, y \in \mathbb{R}$.

(ii) $e^x > 0$ *for all* $x \in \mathbb{R}$ *and* $e^x > 1$ *for all* $x > 0$.

(iii) $x \mapsto e^x$ *is an increasing function, i.e.* $e^x < e^y$ *for any* $x < y$.

(iv) $\lim_{x\to+\infty} e^x = +\infty$ *and* $\lim_{x\to-\infty} e^x = 0$.

The proof of (i) was sketched in last week's lecture. To see (ii), we first see that if $x > 0$,

$$e^x = 1 + x + \frac{x}{2} + \cdots > 1$$

since all the terms we have dropped are positive. To see that $e^x > 0$ for $x \leq 0$, first of all we have $e^0 = 1$, using (i) we have

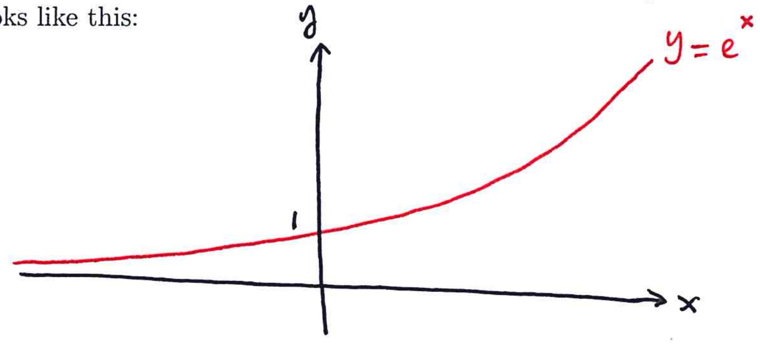
$$e^{-x} = \frac{1}{e^x} > 0$$

for any $x > 0$. To prove (iii), again we use (i) combined with (ii): if $x < y$, then $y = x + h$ for some $h > 0$, therefore by (i)

$$e^y = e^{x+h} = e^x \cdot e^h > e^x \cdot 1 = e^x$$

since $e^h > 1$ by (ii). For (iv), notice that for $x > 0$, we have $e^x > 1 + x$ since we are dropping only positive terms. As $1 + x \to +\infty$ as $x \to +\infty$, we also have $\lim_{x \to +\infty} e^x = +\infty$. The other limit follows from the relationship that $e^{-x} = \frac{1}{e^x}$ for all $x > 0$.

Combining all of these properties above, the graph of the exponential function looks like this:



2 Trigonometric functions

Define two functions $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$ by

$$\sin x := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

$$\cos x := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

It is a nontrivial fact that the two series definitions above agree with the usual definition of sin and cos using right angled triangles in trigonometry where x is the angle measured in radians. Similarly, we also define

$$\tan x := \frac{\sin x}{\cos x}.$$

Example 2.1 *Find the first few terms for a series definition of* $\tan x$.

Solution: Suppose that $\tan x$ has a series expansion like this:

$$\tan x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

By definition, $\tan x \ \cos x = \sin x$, putting it their series expansions, we have

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Expanding the left hand side and collecting like terms, we obtain

$$a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right) x^2 + \left(a_3 - \frac{a_1}{2}\right) x^3 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Compare their coefficients on both sides, we get $a_0 = 0$, $a_1 = 1$, $a_2 - \frac{a_0}{2} = 0$ hence $a_2 = 0$. Also, $a_3 - \frac{a_1}{2} = -\frac{1}{6}$ hence $a_3 = \frac{1}{3}$. Therefore, the first two terms in the series expansion of $\tan x$ is

$$\tan x = x + \frac{x^3}{3} + \cdots.$$

Question: Can you find the general term a_k in the series expansion of $\tan x$?

Proposition 2.2 *We have the following well-known properties for trigonometric functions.*

(i) (Periodicity)

$$\sin(x + 2\pi) = \sin x,$$
$$\cos(x + 2\pi) = \cos x.$$

(ii) $\sin^2 x + \cos^2 x = 1$.

(iii) (Double angle formula)

$$\sin 2x = 2 \sin x \cos x,$$
$$\cos 2x = \cos^2 x - \sin^2 x.$$

(iv) (Half angle formula)

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2},$$
$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}.$$

Note that (ii) is in fact a restatement of Pythagoras' Theorem. We can actually derive (iv) from (iii) (Exercise!) It is also a good exercise to try to "prove" the identities above using only the series definitions.

We can define the following cousins of trigonometric functions by taking reciprocal:

$$\sec x := \frac{1}{\cos x}, \quad \csc x := \frac{1}{\sin x}, \quad \cot x := \frac{1}{\tan x}.$$

Unlike their cousins, these new functions are not defined on the whole real line $\mathbb{R}$. (Question: what are their maximal domain of definition? Can you sketch the graph of all these six trigonometric functions?)

Exercises: Prove the following identities:

(i) $1 + \tan^2 x = \sec^2 x$,

(ii) $1 + \cot^2 x = \csc^2 x$.

3 Hyperbolic functions

We can use the exponential function to define two functions $\sinh : \mathbb{R} \to \mathbb{R}$ and $\cosh : \mathbb{R} \to \mathbb{R}$, called the hyperbolic sine and hyperbolic cosine respectively, by

$$\sinh x := \frac{e^x - e^{-x}}{2},$$
$$\cosh x := \frac{e^x + e^{-x}}{2}.$$

We also define the following analogous to the usual trigonometric functions:

$$\tanh x := \frac{\sinh x}{\cosh x}, \quad \operatorname{sech} x := \frac{1}{\cosh x}, \quad \operatorname{csch} x := \frac{1}{\sinh x}, \quad \coth x := \frac{1}{\tanh x}.$$

Example 3.1 *Find the series expansion of* $\sinh x$ *and* $\cosh x$.

Solution: Using the series expansion of e^x, we see

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots.$$

Adding and subtracting the two formulas above, we easily obtain:

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \text{odd part of } e^x,$$
$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \text{even part of } e^x.$$

In other words, we have

$$e^x = \cosh x + \sinh x.$$

We will see in the next section that there is a similar formula for $\sin x$ and $\cos x$, but that involves the use of "complex numbers".

Proposition 3.2 *The following identities are true.*

(i) $\cosh^2 x - \sinh^2 x = 1$.

(ii) $1 - \tanh^2 x = \operatorname{sech}^2 x$, *and* $\coth^2 x - 1 = \operatorname{csch}^2 x$.

(iii) (Sum to product formula)

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

Exercise: Prove the above proposition starting from the definitions.

Example 3.3 *Show that* $\cosh : \mathbb{R} \to \mathbb{R}$ *is neither 1-1 nor onto.*

Solution: Note that cosh is an "even function", i.e. for any $x \in \mathbb{R}$,

$$\cosh(-x) = \cosh x.$$

Therefore, we have $\cosh(1) = \cosh(-1)$ but $1 \neq -1$. So it is not 1-1.

To see that it is not onto, recall that $e^x > 0$ for all $x\mathbb{R}$, therefore, for all $x \in \mathbb{R}$,

$$\cosh x = \frac{e^x + e^{-x}}{2} > 0.$$

In particular, there is no $x \in \mathbb{R}$ such that $\cosh x = -1$. Hence, it is not onto.

Exercise: What is the range of $\cosh : \mathbb{R} \to \mathbb{R}$?

Exercise: Sketch the graph of all these six hyperbolic functions.

4 Additional topics: Euler's formula

We have all known the fact that $x^2 \geq 0$ for any *real* number x. But what if we assume there is some kind of number i which is a solution the equation $x^2 = -1$. Consider all the expressions

$$z = a + bi$$

where a, b are real numbers. Then, we can formally do arithmetic on these *complex numbers* keeping in mind that $i^2 = -1$. For example,

$$(a+bi)(c+di) = (ac - bd) + (bc + ad)i.$$

Hence, suppose we use the series expansion to evaluate e^{ix}:

$$\begin{aligned} e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x. \end{aligned}$$

We have just proved the *Euler's formula*: for any $x \in \mathbb{R}$,

$$e^{ix} = \cos x + i \sin x.$$

Hence, we have the following formula:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Note how similar it is to the definitions of $\cosh x$ and $\sinh x$.
Exercise: Use these formula to prove that

$$\sin(x + y) = \sin x \cos y + \sin y \cos x.$$

5 Introduction to limits

Given a function f, and some real number a, we want to understand what it means by the symbol

$$\lim_{x \to a} f(x) = L.$$

Intuitively, it means that when x gets "closer and closer" to a, the function value $f(x)$ also gets "closer and closer" to the number L, called the limit of f at $x = a$.

Example 5.1 *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.*

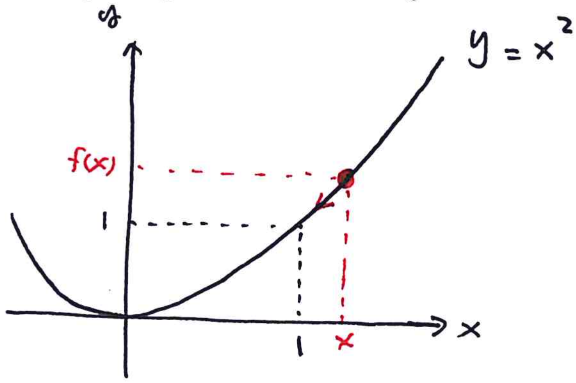
If we draw a table of the values $f(x)$ against x:

x	$f(x)$
1.1	1.21
1.01	1.0201
1.001	1.002001
1.0001	1.00020001
$\vdots$	$\vdots$

Numerically we can see that as x approaches 1, we have $f(x)$ approaches 1 as well. Therefore, we say that

$$\lim_{x \to 1} x^2 = 1.$$

If we look at the graph of the function instead, we see that the height of the points on the graph corresponding to x gets closer to 1 as x gets closer to 1.



For those of you interested, the precise mathematica definition of limit is given below (not required in this class).

Definition 5.2 *We say that* $|im_{x \to a} f(x) = L$ *if for any* $\epsilon > 0$*, there exists* $\delta > 0$ *(depending on* ϵ*) such that*

$$|f(x) - L| < \epsilon$$

for all x *such that* $0 < |x - a| < \delta$.

In some sense, ϵ is the tolerance on the error of the function value $f(x)$ from L, and δ is measuring how "close" we need to be around a to achieve the ϵ-level of error.

6 How to calculate limits I

We will start with some simple examples to illustrate how to calculate limits.

Rule 1: Substitute $x = a$ into $f(x)$ if everything makes sense.

Example 6.1 *(1)* $\lim_{x \to 1} 2x^2 + 3x - 1 = 2(1)^2 + 3(1) - 1 = 4$.

(2) $\lim_{x \to \pi/2} \sin x = \sin \frac{\pi}{2} = 1$.

(3) $\lim_{x\to -1} \frac{x^2-1}{x-1} = \frac{(-1)^2-1}{(-1)-1} = \frac{0}{-2} = 0.$

(4) $\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}(?!).$

In the last example, we see that sometimes we cannot do direct substitution to get the limit. In this situation, we can sometimes apply the next rule.

Rule 2: Simplify the expression first, then substitute.

Example 6.2 *Consider the following:*

(1) $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x+1) = 1+1 = 2.$

(2) $\lim_{x\to 0} \frac{\tan x}{\sin x} = \lim_{x\to 0} \frac{\sin x/\cos x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1.$

(3) $\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x} = \lim_{x\to 4} \frac{(2-\sqrt{x})(2+\sqrt{x})}{(4-x)(2+\sqrt{x})} = \lim_{x\to 4} \frac{4-x}{(4-x)(2+\sqrt{x})}$
$= \lim_{x\to 4} \frac{1}{2+\sqrt{x}} = \frac{1}{2+\sqrt{4}} = \frac{1}{4}.$

The last example is a useful technique called "rationalization".

Note 1: To consider the limit $\lim_{x\to a} f(x)$, the function needs not be defined at $x = a$. See Example 6.2.

Note 2: We can consider the "infinity cases": $x \to \pm\infty$ and/or $L = \pm\infty$.

Example 6.3 *We have*

$$\lim_{\substack{x\to 0\\ x>0}} \frac{1}{x} = +\infty, \qquad \textit{and} \qquad \lim_{x\to +\infty} \frac{1}{x} = 0.$$

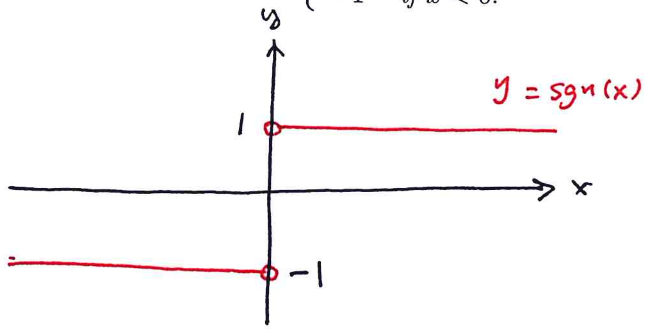
Exercise: What about

$$\lim_{\substack{x\to 0\\ x<0}} \frac{1}{x} \qquad \text{and} \qquad \lim_{x\to -\infty} \frac{1}{x}?$$

Note 3: Limit may not always exists.

Example 6.4 *Show that the limit* $\lim_{x\to 0} f(x)$ *does not exist for the function*

$$f(x) = sgn(x) := \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$



From the graph, it is easy to see that

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = 1, \qquad \text{and} \qquad \lim_{\substack{x\to 0\\ x<0}} f(x) = -1.$$

Since they are not equal to each other, the limit $\lim_{x\to 0} f(x)$ does not exists since we do not know whether we should choose 1 or -1.

In fact, once a limit exists, it must be unique.

Fact: (Uniqueness of limit) If we have

$$\lim_{x\to a} f(x) = L_1 \qquad \text{and} \qquad \lim_{x\to a} f(x) = L_2,$$

then $L_1 = L_2$.